Solutions to Quiz 1, ECED 3300

Problem 1

By definition,

$$\nabla\times\mathbf{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi\\ \partial_r & \partial_\theta & \partial_\phi\\ 0 & 0 & r\sin\theta\times\sin\theta/r^2\end{vmatrix}=\frac{1}{r^3}(2\mathbf{a}_r\cos\theta+\mathbf{a}_\theta\sin\theta).$$

The circulation $\oint d\mathbf{l}\cdot(\nabla\times\mathbf{A})$ depends on the dot product of the curl and $d\mathbf{l}=\mathbf{a}_\phi d\phi$. Since $\mathbf{a}_\theta\cdot\mathbf{a}_\phi=\mathbf{a}_r\cdot\mathbf{a}_\phi=0$, by orthogonality, we readily conclude that $d\mathbf{l}\cdot(\nabla\times\mathbf{A})=0$, implying that $\oint d\mathbf{l}\cdot(\nabla\times\mathbf{A})=0$.

Problem 2

a)Using the Gauss theorem,

$$\oint d\mathbf{s}\cdot\mathbf{r}=\int dv\nabla\cdot\mathbf{r}.$$

The magnitude of divergence is independent of the coordinate system, implying that we can use any coordinate system we please. In the Cartesian coordinates, $\mathbf{r}=x\mathbf{a}_x+y\mathbf{a}_y+z\mathbf{a}_z$. Thus,

$$\nabla\cdot\mathbf{r}=\partial_x(x)+\partial_y(y)+\partial_z(z)=1+1+1=3.$$

Hence,

$$\oint d\mathbf{s}\cdot\mathbf{r}=\int dv3=3v=2\times4\pi/3=4\pi.$$

b) By construction,

$$\mathbf{r}_\perp=\mathbf{r}-\mathbf{a}_z(\mathbf{r}\cdot\mathbf{a}_z)$$

Using the cylindrical coordinates, $\mathbf{r}=\rho\mathbf{a}_\rho+z\mathbf{a}_z$, implying that $\mathbf{r}\cdot\mathbf{a}_z=z$. Finally,

$$\mathbf{r}_\perp=\mathbf{r}-z\mathbf{a}_z=\rho\mathbf{a}_\rho.$$

Problem 3

The fastest increase of the temperature occurs in the direction of the temperature gradient. Hence, the mosquito will fly in the direction determined by

$$\mathbf{a}_m=\frac{\nabla T}{|\nabla T|}.$$

In the cylindrical coordinates, T does not depend on ϕ. It follows that

$$\nabla T = \mathbf{a}_\rho \partial_\rho T + \mathbf{a}_z \partial_z T = -(\rho \mathbf{a}_\rho + 2z\mathbf{a}_z) e^{-\rho^2/2 - z^2}, \quad |\nabla T| = e^{-\rho^2/2-z^2}\sqrt{\rho^2 + 4z^2}.$$

At $(1, 1, 1)$, $\rho = \sqrt{x^2 + y^2} = \sqrt{2}$, $z = 1$, implying that

$$\mathbf{a}_m = -\frac{\sqrt{2}\mathbf{a}_\rho + 2\mathbf{a}_z}{\sqrt{6}} = -\frac{\mathbf{a}_\rho + \sqrt{2}\mathbf{a}_z}{\sqrt{3}}$$